J. Fluid Mech. (1979), *vol.* 95, *part* 1, *pp.* 97–118
Printed in Great Britain

On the effects of thermally insulating boundaries on geostrophic flows in rapidly rotating gases

By FRITZ H. BARK
Department of Mechanics, Royal Institute of Technology, Stockholm, Sweden
AND LENNART S. HULTGREN
Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, Massachusetts

(Received 30 August 1978 and in revised form 29 January 1979)

The effects of thermally insulating boundaries on rapidly and almost rigidly rotating gas flows are examined. It is shown that, on a thermally insulating boundary, all boundary layers disappear to zeroth order and that the geostrophic flow alone satisfies the kinematical boundary condition on such a boundary. The temperature gradient of the geostrophic flow is on a horizontal thermally insulating boundary corrected by a weak Ekman layer of strength $E^{\frac{1}{2}}$ where E is the Ekman number. On a vertical thermally insulating boundary, the temperature gradient of the geostrophic flow is in the general case corrected by $E^{\frac{1}{4}}$ and $E^{\frac{1}{3}}$ Stewartson layers of strengths $E^{\frac{1}{4}}$ and $E^{\frac{1}{3}}$ respectively.

1. Introduction

The study of rapidly rotating gas flows, which is a relatively new field of research in fluid mechanics, is motivated by the technologically important need to understand the flow in gas centrifuges for uranium enrichment. In such centrifuges, weak axial flows, which are superposed on the rigid rotation, are used to control the diffusion process. It turns out that the efficiency of such a centrifuge is very sensitive to the spatial structure of the perturbation flow. The fundamental theoretical study of such flows is the paper by Sakurai & Matsuda (1974). These authors showed that the geostrophic part of the flow is partly governed by a balance of thermal diffusion and viscous diffusion of angular momentum and partly by a non-diffusive balance between the Coriolis force, the pressure gradient and the buoyancy force. The flow is, roughly speaking, in this respect of a similar kind as a geostrophic flow of an axially stratified Boussinesq fluid. The structure of the boundary layers, however, was shown by Sakurai & Matsuda (1974) to be very much the same as that in a rotating flow of a homogeneous fluid. Since the work by Sakurai & Matsuda (1974) was published, a large number of papers dealing with different aspects of rapidly rotating gases have appeared in the literature. The status of the subject up to 1977 is summarized in Soubbaramayer (1977).

In the problem considered by Sakurai & Matsuda (1974), the solid walls of the vessel containing the gas were assumed to be perfectly thermally conducting. The effects of thermally insulating walls were investigated by Matsuda, Hashimoto &

0022-1120/79/4286-7380 $02.00

Takeda (1976), Matsuda & Hashimoto (1976), Hashimoto (1977), Matsuda (1977), Matsuda & Hashimoto (1978) and Matsuda & Takeda (1978). These authors considered a cylindrical container having an insulated cylindrical wall and conducting top and bottom walls or a conducting cylindrical wall and insulating top and bottom walls. The mathematical method used was the classical boundary-layer approach for small Ekman numbers supplemented with the assumption that the gas is very heavy. More precisely, it was assumed that γ, the ratio between the specific heats at constant pressure and volume, is nearly equal to one in the following sense,

$$\gamma - 1 = O(E^{\lambda}), \tag{1.1}$$

where E is the Ekman number. The constant λ is equal to $\frac{1}{2}$ (Matsuda & Hashimoto 1976) or $\frac{1}{4}$ (Matsuda & Hashimoto 1978) if the top and bottom walls are insulating, and equal to $\frac{1}{3}$ if the cylindrical wall is insulating (Matsuda *et al.* 1976; Matsuda & Takeda 1978). The strengths of the boundary layers were assumed to be of order unity. Under these assumptions, it was shown that, among other things, the flow in the Stewartson $E^{\frac{1}{3}}$ layer tends to be suppressed if the cylindrical wall is insulating compared to the case where this wall is conducting. If the top and bottom walls are insulating, the geostrophic axial flow and the flow in the Ekman layers become weak.

The perturbation method used in the aforementioned papers has, as was pointed out by Matsuda (1977), a somewhat restricted range of validity. For instance, the quantity $(\gamma - 1)$ appears in the governing equations combined with the Mach number M for the motion of the centrifuge periphery as $M^2(\gamma - 1)$. This means that the method described above can only be expected to give accurate results for M^2 being of order unity. In practical cases, however, M^2 is usually a large number, which means that $(\gamma - 1)\, M^2$ is of order unity or larger. From a fundamental point of view, it would also be desirable to predict the flow of any gas, not only very heavy ones.

The present paper gives a method for calculating the effect of insulating boundaries on geostrophic flows in rapidly rotating gases without making use of the assumption that the gas is heavy. It is shown that this can be done by requiring that the geostrophic flow satisfies the zeroth order kinematical boundary conditions on thermally insulating boundaries and that the thermal boundary conditions on these boundaries are corrected by weak boundary layers. The latter condition is shown to imply that the Ekman layers are of order $E^{\frac{1}{2}}$ and that the Stewartson $E^{\frac{1}{3}}$ and $E^{\frac{1}{4}}$ layers are of order $E^{\frac{1}{3}}$ and $E^{\frac{1}{4}}$, respectively. An expansion scheme along these lines was briefly outlined by Hashimoto (1977) for a container having insulating top and bottom walls. A similar situation prevails in an axially stratified Boussinesq fluid, where the Ekman layer at an insulated *vertical* wall is of order $E^{\frac{1}{2}}$ (Barcilon & Pedlosky 1966).

The mathematical statement of the problem is given in §2. The flow is assumed to be driven by a symmetric or an antisymmetric differential rotation of the top and bottom walls and an arbitrary temperature distribution on the thermally conducting walls. Numerical examples are presented for the geostrophic part of the flow. In addition to the kinds of containers considered by Matsuda and his co-workers in the papers mentioned above, which in this work are considered in §§4 and 5, the case where every wall is thermally insulating is considered in §3. The results are summarized and compared with earlier results in §6. In order to make the presentation of the results for the geostrophic flow in §§3, 4 and 5 reasonably compact, most of the details regarding the vertical boundary layers are given in appendix 1. Appendix 2 gives some information about the numerical method used.

2. Statement of the problem

Consider an axisymmetric closed container, which consists of a cylindrical wall and flat top and bottom walls. In what follows, the top and bottom walls will frequently be referred to as the horizontal walls and the cylindrical wall will be called the vertical wall. Regarding the thermal properties of the walls, three cases will be considered:

(i) all walls are insulated;
(ii) the horizontal walls are insulated and the vertical wall is conducting;
(iii) the horizontal walls are conducting and the vertical wall is insulated.

The cylindrical vessel contains a viscous, thermally conducting perfect gas of constant temperature T^*_{00} and rotates around its axis of symmetry with the constant angular velocity Ω. The rate of rotation is assumed to be very large in the sense that the density stratification of the gas in the container can be considered to be caused by the centrifugal force field alone. Effects of gravity are thus neglected. The density field ρ^*_{00} in the rigidly rotating gas can then be written as

$$\rho^*_{00}(r^*) = \rho^*_{00}(r^*_0)\exp\left\{\frac{\gamma M^2}{2}\left[\left(\frac{r^*}{r^*_0}\right)^2 - 1\right]\right\}, \tag{2.1}$$

where starred variables are dimensional and

$$r^* = \text{distance from the axis of rotation,} \tag{2.2a}$$

$$r^*_0 = \text{distance from the axis of rotation to the periphery,} \tag{2.2b}$$

$$\gamma = \text{ratio of specific heats at constant pressure and volume,} \tag{2.2c}$$

$$M = \frac{r^*_0\,\Omega}{(\gamma R T^*_{00})^{\frac{1}{2}}},\quad \text{Mach number for the motion of the periphery,} \tag{2.2d}$$

$$R = \text{the gas constant.} \tag{2.2e}$$

The motion of the rigidly rotating gas is assumed to be perturbed by a slight, steady, differential rotation of the horizontal walls. The differential rotation can be either symmetric or antisymmetric. The angular velocity of the top is thus $\Omega + \Delta\Omega$, $\Delta\Omega > 0$, and that of the bottom $\Omega \pm \Delta\Omega$. The Rossby number ϵ is defined as

$$\epsilon = \frac{\Delta\Omega}{\Omega}. \tag{2.2f}$$

ϵ is assumed to be sufficiently small for linear theory to be valid. The height of the container is $2H$. H is chosen as the length scale in the problem. A cylindrical co-ordinate system (r, ϕ, z), where r and z are non-dimensional, will be used. The z axis is taken to coincide with the axis of rotation and the co-ordinate system rotates with the container. The location of the origin is chosen such that the container encloses the volume $|z| \leqslant 1, r \leqslant r_0$.

The non-dimensional aspect ratio of the container is thus r_0. If starred quantities are dimensional, suitable definitions for the non-dimensional dependent variables describing the perturbed flow are

$$\mathbf{u} = (u, v, w) = \frac{\mathbf{u}^*}{\epsilon H \Omega} \quad \text{velocity,} \tag{2.3a}$$

$$p = \frac{p^*}{\epsilon \rho_{00}^*(r_0^*) H^2 \Omega^2} \quad \text{pressure,} \tag{2.3b}$$

$$\rho = \frac{\rho^*}{\epsilon \rho_{00}^*(r_0^*)} \quad \text{density,} \tag{2.3c}$$

$$T = \frac{T^*}{\epsilon T_{00}^*} \quad \text{temperature.} \tag{2.3d}$$

Apart from the factor ϵ^{-1}, the basic density and temperature fields are made non-dimensional as given by (2.3*c*–*d*). Using the notation given above, the linearized non-dimensional governing equations for axisymmetric flow become

$$-2\rho_{00} v - r\rho = -\frac{\partial p}{\partial r} + E\left(\nabla^2 - \frac{1}{r^2}\right) u, \tag{2.4a}$$

$$2\rho_{00} u = E\left(\nabla^2 - \frac{1}{r^2}\right) v, \tag{2.4b}$$

$$0 = -\frac{\partial p}{\partial z} + E\nabla^2 w, \tag{2.4c}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho_{00} u) + \rho_{00}\frac{\partial w}{\partial z} = 0, \tag{2.4d}$$

$$-4\alpha^2 r \rho_{00} u = E\nabla^2 T, \tag{2.4e}$$

$$p = \frac{\sigma(\gamma - 1)}{4\gamma\alpha^2}(\rho + \rho_{00} T), \tag{2.4f}$$

where

$$\rho_{00}(r) = \exp\left[\frac{2\alpha^2\gamma}{\sigma(\gamma - 1)}(r^2 - r_0^2)\right] \tag{2.5}$$

is the non-dimensional basic density field and

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2},$$

$$E = \frac{\mu}{\rho_{00}^*(r_2^*) H^2 \Omega} \quad \text{Ekman number at the periphery,} \tag{2.6a}$$

μ dynamic shear viscosity,

$$\alpha^2 = \frac{\sigma(\gamma - 1) M^2}{4 r_0^2} \tag{2.6b}$$

$$\sigma = \frac{\mu\gamma c_v}{k} \quad \text{Prandtl number,} \tag{2.6c}$$

c_v specific heat at constant volume,

k thermal conductivity.

μ, k, γ and c_v are assumed to be constants. The Ekman number E is assumed to be very small, whereas α^2, σ, γ and r_0 are assumed to be of order unity. In the present problem, viscous effects due to pure dilational motions can be shown to be of higher order and are therefore neglected in (2.4*a*–*c*). If thermally insulating boundaries are denoted by S_i and thermally conducting boundaries by S_c, the system of equations (2.4*a*–*f*) are to be solved subject to the following boundary conditions:

$$\mathbf{u}(r_0, z) = 0, \quad |z| < 1, \tag{2.7a}$$

$$\mathbf{u}(r, 1) = r\mathbf{e}_\phi, \quad 0 \leqslant r \leqslant r_0, \tag{2.7b}$$

$$\mathbf{u}(r, -1) = \pm r\mathbf{e}_\phi, \quad 0 \leqslant r \leqslant r_0, \tag{2.7c}$$

$$\mathbf{n}.\nabla T = 0 \quad \text{on} \quad S_i, \tag{2.7d}$$

$$T = T_c \quad \text{on} \quad S_c, \tag{2.7e}$$

where $\mathbf{n}$ is a unit vector perpendicular to S_i and T_c is the prescribed temperature distribution on S_c.

3. Insulating horizontal and vertical walls

It is assumed that the geostrophic part of the flow, i.e. the flow outside the boundary layers, can be mathematically described by an asymptotic power series in $E^{\frac{1}{2}}$ as follows:

$$(u, v, w, p, \rho, T) = \sum_{n=0}^{N-1} E^{n/2}(u_n, v_n, w_n, p_n, \rho_n, T_n) + O(E^{N/2}). \tag{3.1}$$

By substitution of (3.1) into (2.4*a*–*f*), one finds the following equations:

$$u_0 = u_1 = w_0 = 0, \tag{3.2}$$

$$2\rho_{00} v_0 + r\rho_0 = \frac{\partial p_0}{\partial r}, \tag{3.3a}$$

$$2\rho_{00} u_2 = \left(\nabla^2 - \frac{1}{r^2}\right) v_0, \tag{3.3b}$$

$$\frac{\partial p_0}{\partial z} = 0, \tag{3.3c}$$

$$\frac{\partial w_1}{\partial z} = 0, \tag{3.3d}$$

$$-4\alpha^2 r\rho_{00} u_2 = \nabla^2 T_0, \tag{3.3e}$$

$$p_0 = \frac{\sigma(\gamma - 1)}{4\gamma\alpha^2}(\rho_0 + \rho_{00} T_0). \tag{3.3f}$$

From (3.3*b*) and (3.3*e*) one finds that

$$\nabla^2 T_0 + 2\alpha^2 r\left(\nabla^2 - \frac{1}{r^2}\right) v_0 = 0. \tag{3.4}$$

Equation (3.4) was first given in the literature by Sakurai & Matsuda (1974). From (3.3*a*), (3.3*f*) and (2.5) one can derive the following equation:

$$v_0 = \frac{rT_0}{2} + \frac{1}{2}\frac{dq_0}{dr}, \tag{3.5}$$

where q_0 is defined by

$$q_0 = \frac{p_0}{\rho_{00}}. \tag{3.6}$$

According to (2.5) and (3.3c) q_0 is a function of r only. From (3.4) and (3.5) one finds the following non-homogeneous elliptic partial differential equation for v_0:

$$\frac{\partial}{\partial r}\left[r(1+\alpha^2 r^2)\frac{\partial}{\partial r}\frac{v_0}{r}\right] + (1+\alpha^2 r^2)\frac{\partial^2 v_0}{\partial z^2} = \frac{1}{2}\frac{d}{dr}\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{dq_0}{dr}\right)\right], \tag{3.7}$$

where q_0 depends on v_0 in a so far unknown way. If the horizontal walls are conducting, q_0 can be calculated in terms of the prescribed swirl velocity and temperature on the horizontal walls by analysing the Ekman layers.† After considering the vertical boundary layers at $r = r_0$, one then obtains a well-posed problem for the geostrophic part of the flow [Howard (private communication); Hashimoto 1977]. It should be noted that considerable simplifications occur if the differential rotation of the horizontal walls is antisymmetric because in that case q_0 vanishes according to (3.3c) and (3.6). This case was treated by Sakurai & Matsuda (1974). Also in the present case, q_0 can be determined by studying the Ekman layers and, for reference, some properties of the flow in these layers are therefore recapitulated below. The reader is referred to Sakurai & Matsuda (1974) for details. The boundary layer co-ordinate in the Ekman layers is defined as follows:

$$\zeta = \frac{1 \mp z}{E^{\frac{1}{2}}}, \tag{3.8a}$$

where the minus sign refers to the top and the plus sign to the bottom. The dependent variables in the Ekman layers are denoted by a caret and are assumed to possess expansions of the form

$$(\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{\rho}, \hat{T}) = \sum_{n=0}^{N-1} E^{n/2}(\hat{u}_n, \hat{v}_n, \hat{w}_n, \hat{p}_n, \hat{\rho}_n, \hat{T}_n) + O(E^{N/2}), \tag{3.8b}$$

where $\hat{u}_n, \hat{v}_n$ etc. are assumed to be functions of r and ζ. The following relations can then be shown to hold in the Ekman layers:

$$\hat{w}_0 = \hat{p}_0 = \hat{p}_1 = 0, \tag{3.9}$$

$$\hat{u}_0 + i\kappa^2 \hat{v}_0 = i\kappa^2 V_0 \exp\left[-(1+i)\kappa\rho_{00}^{\frac{1}{2}}\zeta\right], \tag{3.10a}$$

$$\hat{T}_0 + 2\alpha^2 r \hat{v}_0 = 0, \tag{3.10b}$$

$$\hat{w}_1 = -\frac{1}{r\rho_{00}}\frac{\partial}{\partial r}\left(r\rho_{00}\int_{\infty}^{\zeta}\hat{u}_0(r,\zeta')\,d\zeta'\right), \tag{3.10c}$$

where $\kappa = (1+\alpha^2 r^2)^{\frac{1}{2}}$,

and V_0 is the Ekman layer swirl velocity evaluated at $\zeta = 0$. It is readily shown that order of magnitude consistency requires that V_0 is a real function of r. From (3.10a–b) one can show that

$$\frac{\partial T_0}{\partial \zeta}(r, 0) = 2\alpha^2 \kappa \rho_{00}^{\frac{1}{2}} r V_0, \tag{3.11}$$

which means that an Ekman layer of order unity is impossible because there would then be an axial heat flux of order $E^{-\frac{1}{2}}$ at the horizontal walls instead of no axial heat

† This expression is given in §5.

flux as required by (2.7*d*). The geostrophic flow consequently has to fulfil the zeroth order kinematical boundary conditions (2.7*b*–*c*) on the horizontal wall by itself. The thermal boundary condition (2.7*d*) will be taken care of by Ekman layers of order $E^{\frac{1}{2}}$. The indices of the lowest-order Ekman layer quantities in (3.8*b*) are therefore increased by one in what follows. A similar situation was found by Barcilon & Pedlosky (1966) for *vertical* Ekman layers in an axially stratified Boussinesq fluid. In the present case, it is readily shown that the weak horizontal Ekman layers will drive a geostrophic flow of order $E^{\frac{1}{2}}$. Furthermore, the geostrophic axial velocity will be of order E. According to (2.7*b*–*c*) and (2.7*d*), the following relations must hold at the horizontal walls:

$$w_2(r, \pm 1) + \hat{w}_2(r, 0) = 0, \tag{3.12a}$$

$$\frac{\partial T_0}{\partial z}(r, \pm 1) + \frac{\partial \hat{T}_1}{\partial \zeta}(r, 0) = 0, \tag{3.12b}$$

where no notational distinction has been made between the Ekman layers at the top and the bottom walls. From (3.12*a*), (3.10*a*–*c*) and (3.12*b*) one finds that

$$w_2(r, \pm 1) = \frac{1}{4\alpha^2 r \rho_{00}} \frac{\partial^2 T_0}{\partial r \partial z}(r, \pm 1). \tag{3.13}$$

Next, consider the equation of continuity for the lowest-order geostrophic flow

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho_{00} u_2) + \rho_{00}\frac{\partial w_2}{\partial z} = 0. \tag{3.14}$$

For notational simplicity, vertical averages will in what follows be denoted by $\langle\,\rangle$, e.g.,

$$\langle u_2 \rangle = \frac{1}{2}\int_{-1}^{1} u_2\, dz.$$

Integration of (3.14) with respect to r and z gives

$$r\rho_{00}\langle u_2\rangle + \int_0^r r'\rho_{00}(r')\,[w_2(r', 1) - w_2(r', -1)]\,dr' = 0. \tag{3.15}$$

Substitution of (3.13) and the vertical average of (3.3*b*) into (3.15) gives

$$2\alpha^2 r\left[\frac{d}{dr}\frac{1}{r}\frac{d}{dr}\, r\langle v_0\rangle + \frac{\partial v_0}{\partial z}(r, 1) - \frac{\partial v_0}{\partial z}(r, -1)\right] + \frac{\partial T_0}{\partial z}(r, 1) - \frac{\partial T_0}{\partial z}(r, -1) = 0. \tag{3.16}$$

If one takes the vertical average of (3.4) and combines the result with (3.16), one finds the following relation:

$$\frac{d}{dr}r\frac{d}{dr}\langle T_0\rangle = 0, \tag{3.17}$$

whose regular solution is

$$\langle T_0\rangle = 0, \tag{3.18}$$

where the constant of integration has been set equal to zero, which can be done without loss of generality by choosing a suitable reference temperature. From (3.18) and (3.5) one then finds that

$$\frac{dq_0}{dr} = 2\langle v_0\rangle, \tag{3.19a}$$

$$T_0 = \frac{2}{r}(v_0 - \langle v_0\rangle). \tag{3.19b}$$

The geostrophic flow can then, according to (3.7) and (3.19*a*), be calculated from the following integro-partial differential equation:

$$\frac{\partial}{\partial r}\left[r(1+\alpha^2 r^2)\frac{\partial}{\partial r}\frac{v_0}{r}\right]+(1+\alpha^2 r^2)\frac{\partial^2 v_0}{\partial z^2}=\frac{d}{dr}\left(r\frac{d}{dr}\frac{\langle v_0\rangle}{r}\right). \tag{3.20}$$

In order to determine the boundary conditions at the vertical wall for the solution of (3.20), the Stewartson boundary layers of thickness $E^{\frac{1}{4}}$ and $E^{\frac{1}{3}}$ have to be considered. The $E^{\frac{1}{4}}$ layer occurs only if the differential rotation is symmetric. However, as the derivation of the boundary conditions at the vertical wall follows very much along the same lines as that for the horizontal walls, only a brief discussion will be given in this section. Some further details are given in appendix 1.

In both the $E^{\frac{1}{4}}$ and $E^{\frac{1}{3}}$ layers, the following relation holds (cf. 3.10*b*):

$$\tilde{T}+2\alpha^2 r_0\tilde{v}=0, \tag{3.21}$$

where tildes denote quantities in either one of the layers (e.g. see Bark & Bark 1976). As in the Ekman layer case, (3.21) can be shown to imply that the geostrophic flow has to fulfil the zeroth order kinematic boundary condition (2.7 *a*) also at the vertical wall and that higher-order vertical boundary layers have to take care of the thermal boundary condition (2.7 *d*). As in the homogeneous fluid case, the vertical averages of the geostrophic flow are to be corrected by the $E^{\frac{1}{4}}$ layer and the remaining parts by the $E^{\frac{1}{3}}$ layer. However, in the present case $\langle T_0\rangle$ is zero [see (3.18)]. This means that there will be no $E^{\frac{1}{4}}$ layer of order $E^{\frac{1}{4}}$. There will, though, be a $E^{\frac{1}{4}}$ layer of order $E^{\frac{1}{2}}$ to adjust the vertically averaged geostrophic radial mass flux at the periphery. The $E^{\frac{1}{3}}$ layer will be of order $E^{\frac{1}{3}}$ and corrects $\partial T_0/\partial r$.

The solution of (3.20) thus has to fulfil the following boundary conditions:

$$v_0(r,1)=r, \quad 0\leqslant r\leqslant r_0, \tag{3.22a}$$

$$v_0(r,-1)=\pm r, \quad 0\leqslant r\leqslant r_0, \tag{3.22b}$$

$$v_0(r_0,z)=0, \quad |z|<1. \tag{3.22c}$$

Some numerical results are shown in figures 1–7. The numerical method is briefly described in appendix 2. In all the numerical examples given in this work, the values of γ and σ are those for UF_6 at room temperature, i.e. $\gamma = 1{\cdot}067$, $\sigma = 0{\cdot}95$.

It will be shown in §4 that the solutions shown in figures 1–7 are also valid if the vertical wall is conducting and isothermal. Figure 1 shows the vertically averaged local angular velocity field $\langle v_0\rangle/r$ for the symmetric case for different values of the Mach number. The results in figure 1 should be compared with the corresponding homogeneous fluid case where the geostrophic flow is everywhere a rigid rotation with the same angular velocity as the top and bottom walls (Stewartson 1957). Figure 1 shows that, for small Mach numbers, the geostrophic flow is very nearly the same as in the homogeneous fluid case apart from the immediate neighbourhood of the periphery. The reason for the finite difference near the periphery between the zero Mach number limit, i.e. the homogeneous fluid case, and the small but non-zero Mach number case is, of course, due to the completely different structure of the $E^{\frac{1}{4}}$ layer in the two cases. Figure 2 shows the local angular velocity field in the symmetric case for a low Mach number.

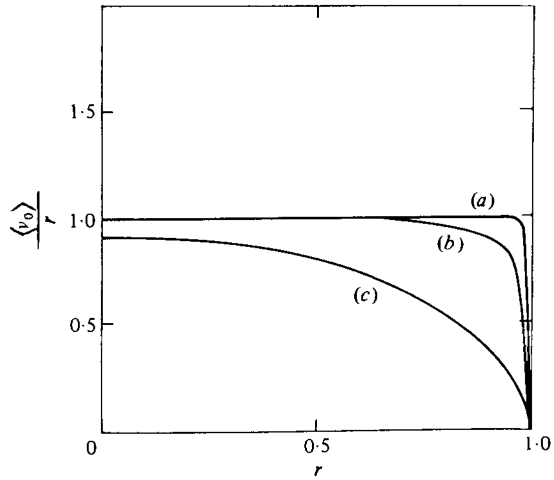

FIGURE 1. Insulated horizontal walls. $\langle v_0 \rangle / r$ for the symmetric case. (a) $M = 0{\cdot}1$, (b) $M = 1$, (c) $M = 5$.

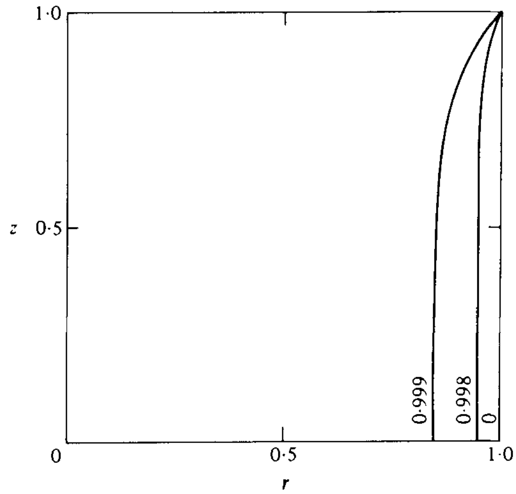

FIGURE 2. Insulated horizontal walls. v_0/r for the symmetric case. $M = 0{\cdot}1$.

Because the container is thermally insulated and viscous dissipation is neglected, temperature differences between different parts of the gas can only be produced by local compression or expansion, which occurs for gas particles moving radially in the basic pressure field. It should be pointed out that the temperature field is also affected by thermal diffusion. Figures 3 and 4 show, for the symmetric case, a comparison between the temperature fields for $M = 5$ and $M = 10$. There is, in both cases, a clockwise meridional circulation. Near $z = 0$ the gas moves radially inwards, whereby expansion and cooling takes place. The opposite process occurs near the top. The temperature differences are, as expected, larger for the $M = 10$ case. The corresponding graphs of the local angular velocity field v_0/r are shown in figures 5 and 6, from which it can be seen that the deviation from rigid rotation is larger for the $M = 10$ case. Figure 7 shows v_0/r for $M = 10$ for the antisymmetric case. It was found from numerical experiments that the flow field depends rather weakly on the Mach number in the antisymmetric case whereas this is not so for the symmetric case.

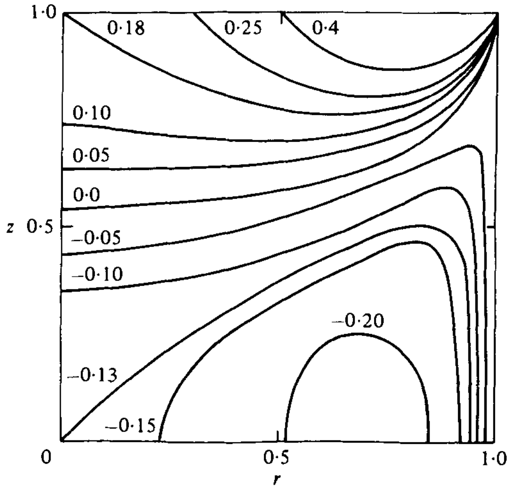

FIGURE 3. Insulated horizontal walls. T_0 for the symmetric case. $M = 5$.

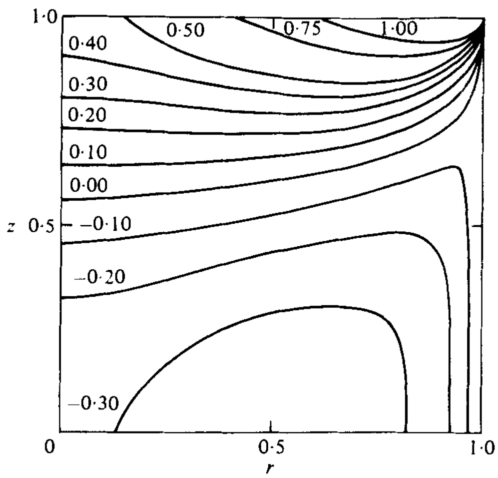

FIGURE 4. Insulated horizontal walls. T_0 for the symmetric case. $M = 10$.

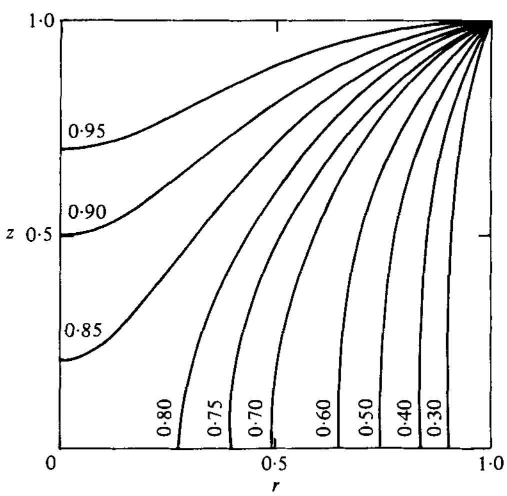

FIGURE 5. Insulated horizontal walls. v_0/r for the symmetric case. $M = 5$.

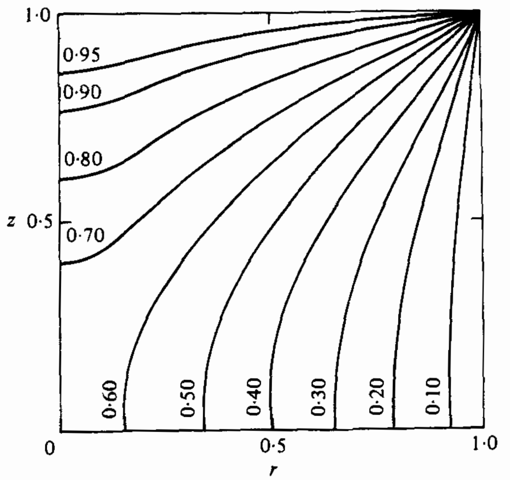

FIGURE 6. Insulated horizontal walls. v_0/r for the symmetric case. $M = 10$.

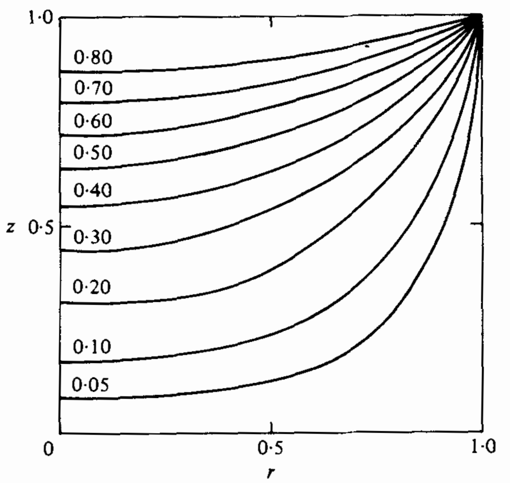

FIGURE 7. Insulated horizontal walls. v_0/r for the antisymmetric case. $M = 10$.

4. Insulated horizontal and conducting vertical walls

In this case the Ekman layers have the same structure as in the case dealt with in § 3. This means that the equation for the geostrophic flow is given by (3.20) and the boundary conditions at the horizontal surfaces by (3.22a–b). At the vertical wall, however, the boundary condition for the geostrophic flow will be different because the perturbation temperature $T_c(z)$ is prescribed instead of the heat flux. It is shown in appendix 1 that there cannot be an $E^{\frac{1}{4}}$ layer of order unity in this case. The reason for this is that the order unity Ekman layer extensions, which are a pre-requisite for the existence of an order unity $E^{\frac{1}{4}}$ layer, cannot occur for the same reason as that prohibiting the order unity Ekman layers. Thus, a situation where $\langle T_0 \rangle \neq \langle T_c \rangle$ cannot occur, and one must consequently require that (cf. 3.18)

$$\langle T_c \rangle = 0 \tag{4.1}$$

in order to have a well-posed problem. Because of the absence of an order unity $E^{\frac{1}{4}}$ layer, the geostrophic flow must satisfy

$$\langle v_0 \rangle = 0, \quad r = r_0. \tag{4.2}$$

As in the case discussed in § 3, one can show that there will be an $E^{\frac{1}{4}}$ layer of order $E^{\frac{1}{2}}$. The $E^{\frac{1}{3}}$ layer will, however, be of order unity and this implies that the geostrophic flow must satisfy

$$v(r_0, z) = \frac{r_0 T_c}{2(1+\alpha^2 r_0^2)}. \tag{4.3}$$

Both the weak $E^{\frac{1}{4}}$ layer and the $E^{\frac{1}{3}}$ layer are briefly discussed in appendix 1. The boundary-value problem to be solved in the present case is thus equation (3.20) and the boundary conditions (3.22*a–b*) and (4.3). It should be noted that the boundary condition (4.3) becomes the same as (3.22*c*) if T_c is zero. In this special case a conducting vertical wall is thus equivalent to a thermally insulating vertical wall as far as the geostrophic flow is concerned as do figures 1–7 show the geostrophic flow also in the present problem. The flow in the vertical boundary layers, though, will be different, as is discussed in some detail in appendix 1. No calculations were carried out for cases where T_c is non-zero.

5. Conducting horizontal and insulating vertical walls

The Ekman layers are in this case of order unity and the equation for the geostrophic flow is, as expected, the same as in the case where all boundaries are conducting, i.e.

$$\frac{\partial}{\partial r}\left[r(1+\alpha^2 r^2)\frac{\partial}{\partial r}\frac{v_0}{r}\right] + (1+\alpha^2 r^2)\frac{\partial^2 v_0}{\partial z^2} = \frac{1}{2}\frac{d}{dr}\left\{r\frac{d}{dr}\left(\frac{v_+ + v_-}{r} - \frac{T_+ + T_-}{2}\right)\right\}, \tag{5.1}$$

where v_+, T_+ and v_-, T_- are the prescribed swirl velocities and temperatures at the top and bottom walls respectively. Equation (5.1) has for the general case been derived independently, by Howard (private communication) and Hashimoto (1977) and, for the case where the forcing is antisymmetric, by Sakurai & Matsuda (1974). The relation between the geostrophic swirl velocity and temperature fields is given by (3.5) where q_0 in this case is given by

$$\frac{dq_0}{dr} = v_+ + v_- - \frac{r}{2}(T_+ + T_-).$$

It can also be shown that the boundary conditions for the geostrophic flow on the horizontal walls are given by

$$v_0(r, \pm 1) = v_\pm \pm \frac{v_+ - v_- - \frac{1}{2}r\,(T_+ - T_-)}{2(1+\alpha^2 r^2)}. \tag{5.2}$$

The boundary condition at the insulated vertical wall is, of course, given by (3.22*c*).

It should be noted that the lowest-order Ekman layers disappear if the forcing is symmetric (cf. 5.2). A similar situation occurs in the corresponding homogeneous fluid case (Stewartson 1957), where both the Ekman layers and the geostrophic flow disappear and the flow consists of vertical boundary layers only. In the present case, however, there will still be a geostrophic flow.

Because the vertically averaged geostrophic temperature field in general is non-zero, there is an $E^{\frac{1}{4}}$ layer of order $E^{\frac{1}{4}}$ at the vertical boundary as well as an $E^{\frac{1}{3}}$ layer of order $E^{\frac{1}{4}}$. It is shown in appendix 1 that there is also a stronger $E^{\frac{1}{3}}$ layer of order $E^{\frac{1}{6}}$ if the forcing is antisymmetric.

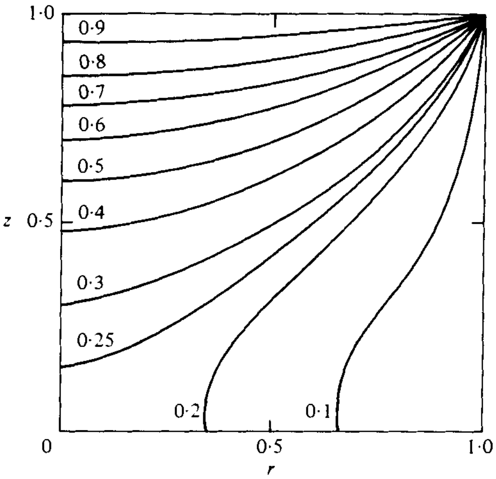

FIGURE 8. Conducting horizontal walls and an insulated vertical wall. v_0/r for the symmetric case. $M = 10$.

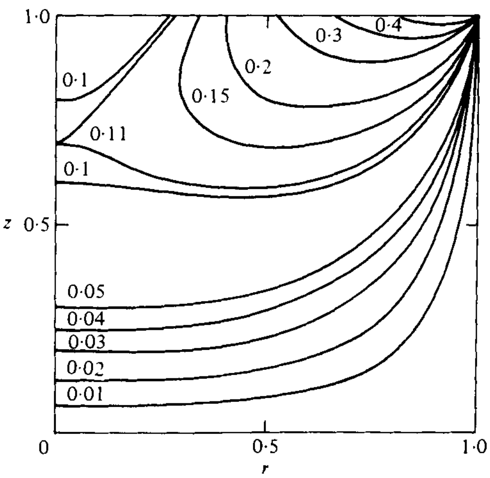

FIGURE 9. Conducting horizontal walls and an insulated vertical wall. v_0/r for the antisymmetric case. $M = 10$.

Some numerical results are shown in figures 8 and 9. Numerical experiments showed that the local angular velocity field v_0/r depends rather weakly on the Mach number in the symmetric case whereas this was not so for the antisymmetric case. This can be interpreted as a manifestation of the fact that the Ekman suction, which depends strongly on the Mach number, disappears to lowest order in the symmetric case.

6. Comparison with earlier work and conclusions

A detailed comparison between the methods and results given by Matsuda *et al.* (1976), Matsuda & Hashimoto (1976, 1978) and Matsuda & Takeda (1978) and those given in the present work would be rather lengthy. Only a brief discussion of some main differences and similarities will therefore be given.

The method given in the present work can be used for any gas for any sufficiently

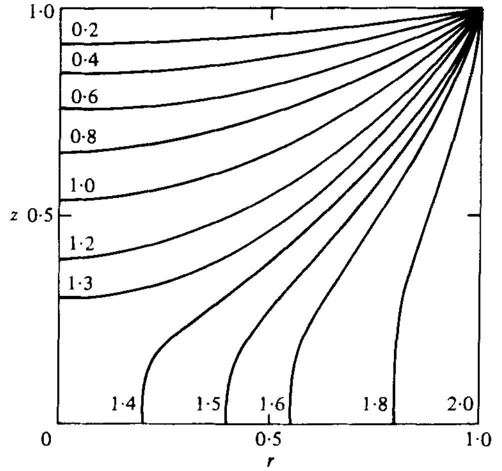

FIGURE 10. Conducting horizontal walls and an insulated vertical wall. T_0 for the symmetric case. $M = 0{\cdot}1$.

small Ekman number whereas the earlier methods rely on a relation of the form (1.1) between the ratio of the specific heats at constant pressure and volume and the Ekman number. The present method can also be used for any finite Mach number whereas the earlier methods are low Mach number or, which turns out to be equivalent, heavy gas approximations. In the paper by Matsuda *et al.* (1976), who considered an antisymmetric, thermally driven flow in a container having conducting horizontal walls and an insulating vertical wall, the limit where the Ekman number approaches zero, and $\gamma - 1$ remains small but fixed, was calculated as a special case ($\alpha = \infty$ in their notation). That solution (see Matsuda *et al.* 1976, figure 3*a* on p. 392) has very much the same character as the solutions given in this work and the lowest order $E^{\frac{1}{3}}$ layer was indeed found to disappear. However, no indication of the presence of a weaker $E^{\frac{1}{3}}$ layer was given and it is therefore not entirely obvious how the geostrophic flow was calculated because it cannot fulfil both the kinematical and the thermal boundary conditions at the vertical wall.

It should be pointed out that the earlier results showed that the boundary layer flows, although these were assumed to be of order unity, became weaker near insulating surfaces compared to those occurring near conducting surfaces. This trend is further amplified in the present results, where all boundary layers near insulating surfaces are of higher order.

A quantitative comparison between results from the two types of expansion schemes can be made from figure 10 in this work and figure 3*a* on p. 454 in the paper by Matsuda & Takeda (1978). Both graphs show the isotherms in a symmetric low-Mach-number geostrophic flow having conducting horizontal walls and an insulating vertical wall, i.e. the kind of container dealt with in § 5. In this case the vertical wall rotates faster than the horizontal walls. There are obviously differences of order unity between the two graphs. The main reason for these differences is the different boundary conditions at the vertical wall, which introduces a singularity in the corner in the present solution whereas no such singularity occurs in the solution given by Matsuda & Takeda (1978). Furthermore, the angle between the isotherms and the vertical wall can be non-zero in the solution given by Matsuda & Takeda (1978). In the present solution the vertical wall coincides with the $T = 2$ isotherm.

Another consequence of the fact that the Ekman suction is of order E if the horizontal boundaries are insulating is that there will be no response on the $\Omega^{-1}E^{-\frac{1}{2}}$ time scale of the gas in the interior parts of the container to a sudden change of the rotation rate of the container. Such a response was calculated by Bark, Meijer & Cohen (1978) for a container having conducting boundaries. If the horizontal boundaries are insulating, the main part of the gas will respond only on the diffusive time scale. Work on this problem is in progress and will be reported in the future.

The authors are grateful to Professors Louis N. Howard, Willem V. R. Malkus and Mårten T. Landahl for many illuminating discussions of the problems considered in this paper.

Appendix 1. Vertical boundary layers

In this appendix some details of the $E^{\frac{1}{3}}$ layer will be given. The $E^{\frac{1}{4}}$ layer will be discussed very briefly at the end. For the $E^{\frac{1}{3}}$ layer, only the formulation of the boundary-value problems to be solved will be given. The solution of these boundary-value problems by separation of variables is straightforward and can be carried out in the manner given by, for example, Sakurai & Matsuda (1974).

It is assumed in this work that the variation of the basic density field is negligible within the $E^{\frac{1}{4}}$ layer. This means that $\rho_{00} = 1$ in both the $E^{\frac{1}{3}}$ and $E^{\frac{1}{4}}$ layers. In the $E^{\frac{1}{3}}$ layer, a stretched co-ordinate ξ, defined by

$$\xi = \frac{r_0 - r}{E^{\frac{1}{3}}}, \tag{A 1}$$

will be used. It is furthermore assumed that the dependent variables in this layer, denoted by tildes, possess asymptotic expansions of the form

$$(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}, \tilde{\rho}, \tilde{T}) = E^s \sum_{n=0}^{N-1} E^{n/3}(\tilde{u}_n, \tilde{v}_n, \tilde{w}_n, \tilde{p}_n, \tilde{\rho}_n, \tilde{T}_n) + O(E^{N/3}), \tag{A 2}$$

where $\tilde{u}_n, \tilde{v}_n$ etc. are functions of ξ and z. The factor E^s serves as a gauge function and is introduced for notational convenience. From (2.4*a*–*f*), (A 1) and (A 2) one finds the following lowest-order equations:

$$\tilde{u}_0 = \tilde{p}_0 = 0, \tag{A 3}$$

$$2\tilde{v}_0 + r_0 \tilde{\rho}_0 + \frac{\partial \tilde{p}_1}{\partial \xi} = 0, \tag{A 4a}$$

$$2\tilde{u}_1 = \frac{\partial^2 \tilde{v}_0}{\partial \xi^2}, \tag{A 4b}$$

$$\frac{\partial \tilde{p}_1}{\partial z} = \frac{\partial^2 \tilde{w}_0}{\partial \xi^2}, \tag{A 4c}$$

$$-\frac{\partial \tilde{u}_1}{\partial \xi} + \frac{\partial \tilde{w}_0}{\partial z} = 0, \tag{A 4d}$$

$$-4\alpha^2 r_0 \tilde{u}_1 = \frac{\partial^2 \tilde{T}_0}{\partial \xi^2}, \tag{A 4e}$$

$$\tilde{\rho}_0 + \tilde{T}_0 = 0. \tag{A 4f}$$

From (A 4b) and (A.4e) one finds, after two integrations with respect to ξ, the following relation:

$$\tilde{T}_0 + 2\alpha^2 r_0 \tilde{v}_0 = 0. \qquad \text{(A 5)}$$

It turns out to be convenient to introduce a stream function $\tilde{\phi}_0(\xi, z)$, which is defined by

$$\tilde{u}_1 = \frac{\partial \tilde{\phi}_0}{\partial z}, \qquad \text{(A 6a)}$$

$$\tilde{w}_0 = \frac{\partial \tilde{\phi}_0}{\partial \xi}. \qquad \text{(A 6b)}$$

It can be shown from (A 4a–f), (A 5) and (A 6a–b), after some algebra, that $\tilde{\phi}_0$ satisfies the following equation:

$$\frac{\partial^6 \tilde{\phi}_0}{\partial \xi^6} + 4(1 + \alpha^2 r_0^2) \frac{\partial^2 \tilde{\phi}_0}{\partial z^2} = 0. \qquad \text{(A 7)}$$

(A 5) and (A 7) were first given in the literature by Sakurai & Matsuda (1974). Next, the boundary conditions to be satisfied by the solution of (A 7) have to be formulated. For the case dealt with in §3, the $E^{\frac{1}{3}}$ layer has to correct the radial temperature gradient of the geostrophic flow at the periphery. Thus, $s = \frac{1}{3}$ in this case. From (A 4b), (A 5) and (A 6a) one finds that

$$\frac{\partial \tilde{T}_0}{\partial \xi} = 4\alpha^2 r_0 \int_{\xi}^{\infty} \frac{\partial \tilde{\phi}_0}{\partial z} d\xi, \qquad \text{(A 8)}$$

which gives the following boundary condition for $\tilde{\phi}_0$:

$$\frac{\partial T_0}{\partial r}(r_0, z) - 4\alpha^2 r_0 \int_0^{\infty} \frac{\partial \tilde{\phi}_0}{\partial z}(\xi, z)\, d\xi = 0, \quad |z| < 1, \qquad \text{(A 9)}$$

where T_0 is the geostrophic temperature field. It should be remembered that $\langle T_0 \rangle = 0$ in this case [see (3.18)]. A simple order-of-magnitude analysis shows that a consistent expansion of the form given by (A 2) can be constructed if the meridional flow in the $E^{\frac{1}{3}}$ layer fulfils the kinematical condition at the periphery, i.e.

$$\tilde{\phi}_0(0, z) = 0, \quad |z| < 1, \qquad \text{(A 10a)}$$

$$\frac{\partial \tilde{\phi}_0}{\partial \xi}(0, z) = 0, \quad |z| < 1, \qquad \text{(A 10b)}$$

(A 10a) also means that, to order $E^{\frac{1}{3}}$, there is no net mass flux to be communicated by the $E^{\frac{1}{3}}$ layer between the top and bottom Ekman layers. Because the swirl velocity in the $E^{\frac{1}{3}}$ layer is non-zero at $\xi = 0$, a geostrophic flow of order $E^{\frac{1}{3}}$ will be driven by the boundary layer. The solution of (A 7) can only fulfil one boundary condition at $z = \pm 1$. It can be shown that, by considering the orders of magnitude of the dependent variables in the available correction fields, the axial velocity $\tilde{w}_0$ has to fulfil the kinematical boundary condition, thus

$$\tilde{\phi}_0(\xi, \pm 1) = 0. \qquad \text{(A 11)}$$

As is well known, (A 11) holds also if every boundary is conducting (e.g. see Sakurai & Matsuda 1974) as well as in the homogeneous fluid problem (Stewartson 1957). The remaining boundary condition is the requirement of exponential decay of all the dependent variables for large values of ξ, i.e.

$$\lim_{\xi \to \infty} \tilde{\phi}_0(\xi, z) = 0. \qquad \text{(A 12)}$$

The formulation of the boundary-value problem for the $E^{\frac{1}{3}}$ layer for the case dealt with in § 3 is now complete to lowest order.

It should be noted that the thermal boundary condition (2.7d) and the no-slip condition (2.7b–c) for the horizontal velocity field are not satisfied by the $E^{\frac{1}{3}}$ layer solution at $z = \pm 1$. Because the radial velocity is of higher order than the swirl velocity (see A 3), it will be omitted from the discussion. It can be shown that the thermal boundary condition can be handled by an Ekman layer extension field of the size $E^{\frac{1}{2}} \times E^{\frac{1}{3}}$ and having the strength $E^{\frac{1}{6}}$. The form of this correction field is, apart from a parametric dependence on ξ, the same as that of the Ekman layer discussed in § 3 and will therefore not be given explicitly. The swirl velocity in the $E^{\frac{1}{3}}$ layer is corrected at $z = \pm 1$ by square-shaped fields of the size $E^{\frac{1}{3}} \times E^{\frac{1}{3}}$ in the corners between the horizontal walls and the vertical wall. Correction fields of this kind were discovered by Matsuda & Hashimoto (1976) and Matsuda *et al.* (1976). To describe the flow in these regions, another boundary-layer co-ordinate χ, defined by

$$\chi = \frac{1 \pm z}{E^{\frac{1}{3}}}, \tag{A 13}$$

is introduced. The dependent variables, denoted by double tildes, are assumed to possess expansions of the form

$$(\tilde{\tilde{u}}, \tilde{\tilde{v}}, \tilde{\tilde{w}}, \tilde{\tilde{p}}, \tilde{\tilde{\rho}}, \tilde{\tilde{T}}) = E^{s} \sum_{n=0}^{N-1} E^{n/3}(\tilde{\tilde{u}}_n, \tilde{\tilde{v}}_n, \tilde{\tilde{w}}_n, \tilde{\tilde{p}}_n, \tilde{\tilde{\rho}}_n, \tilde{\tilde{T}}_n) + O(E^{N/3}), \tag{A 14}$$

where $\tilde{\tilde{u}}_n$, $\tilde{\tilde{v}}_n$ etc. are assumed to be functions of ξ and χ. Substitution of (A 14), (A 1) and (A 13) into (2.4a–f) gives the following equations for the lowest-order quantities:

$$\tilde{\tilde{u}}_0 = \tilde{\tilde{w}}_0 = \tilde{\tilde{p}}_0 = \tilde{\tilde{p}}_1 = 0, \tag{A 15a}$$

$$\tilde{\tilde{v}}_0 = \frac{r_0 \tilde{\tilde{T}}_0}{2}, \tag{A 15b}$$

$$\frac{\partial^2 \tilde{\tilde{v}}_0}{\partial \xi^2} + \frac{\partial^2 \tilde{\tilde{v}}_0}{\partial \chi^2} = 0. \tag{A 15c}$$

It is readily shown from (3.3a–f) that the force balance in the $E^{\frac{1}{3}} \times E^{\frac{1}{3}}$ regions is the same as that in a geostrophic flow having a negligible pressure. At the vertical wall, one finds that $\tilde{\tilde{T}}_0$ has to fulfil the thermal boundary condition (2.7d) because otherwise the higher-order corrections would have to take care of a heat source, having a strength of order unity, which would lead to an inconsistent expansion. The solution of (A 15b–c) thus has to satisfy the following boundary condition:

$$\tilde{\tilde{v}}_0(\xi, 0) + \tilde{v}_0(\xi, \pm 1) = 0, \tag{A 16a}$$

$$\frac{\partial \tilde{\tilde{T}}_0}{\partial \xi}(0, \chi) = 0, \tag{A 16b}$$

$$\lim_{\substack{\chi \text{ fixed} \\ \xi \to \infty}} \tilde{\tilde{v}}_0 = \lim_{\substack{\xi \text{ fixed} \\ \chi \to \infty}} \tilde{\tilde{v}}_0 = 0. \tag{A 16c}$$

It should be noted that the solution of (A 15b–c) and (A 16a–c) in general does not fulfil the thermal boundary condition (2.7d) at $\chi = 0$. However, this discrepancy can be removed by Ekman layer extensions of order $E^{\frac{1}{2}}$. The analysis of the $E^{\frac{1}{3}}$ layer is terminated at this point. The remaining lowest-order boundary conditions to be fulfilled by the higher approximations are the non-zero values of $\tilde{\tilde{v}}_0$ at the vertical wall near the top and bottom. From the point of view of the geostrophic flow, this appears as concentrated sources of angular momentum of strength $E^{\frac{2}{3}}$ in the corners. It is indeed very reasonable to assume that the response of the gas to such sources will be of higher order.

For the case discussed in §4, the $E^{\frac{1}{3}}$ layer has to correct the part of the geostrophic swirl velocity field which has zero vertical average, and the geostrophic temperature field, whose vertical average in this case is identically zero. This means that $s = 0$ in this case. The following boundary conditions must consequently be fulfilled:

$$\tilde{v}_0(0,z) + v_0(r_0,z) - \langle v_0(r_0,z)\rangle = 0, \quad |z| < 1, \tag{A 17a}$$

$$\tilde{T}_0(0,z) + T_0(r_0,z) = T_c(z), \quad |z| < 1, \tag{A 17b}$$

where T_c is the prescribed temperature at the periphery. From (A 5), (A 17a–b), (3.19b) and (4.5) one finds that

$$\tilde{T}_0(0,z) = \frac{\alpha^2 r_0^2}{1+\alpha^2 r_0^2} T_c(z), \quad |z| < 1. \tag{A 18}$$

From (A 18), (A 17a) and (A 5) one can then readily derive (4.3). (A 18) can, according to (A 8), be expressed in terms of $\tilde{\phi}_0$ as follows:

$$\int_0^\infty \left\{ \int_\xi^\infty \frac{\partial \phi}{\partial z}(\xi', z)\, d\xi' \right\} d\xi = \frac{r_0 T_c(z)}{4(1+\alpha^2 r_0^2)}. \tag{A 19}$$

Also in this case the $E^{\frac{1}{3}}$ layer solution must satisfy (A 10a–b), (A 11) and (A 12). The formulation of the boundary-value problem for the $E^{\frac{1}{3}}$ layer for the case discussed in §4 is now complete to lowest order. It can be shown that Ekman layer extensions and $E^{\frac{1}{3}} \times E^{\frac{1}{3}}$ regions will appear in a similar way to the previous case.

An $E^{\frac{1}{3}}$ layer resembling that appearing in the case discussed in §3 will appear also in the case discussed in §5 although the geostrophic flow, whose temperature gradient is to be corrected by the $E^{\frac{1}{3}}$ layer, is different. However, in the antisymmetric case discussed in §5, there is also a stronger $E^{\frac{1}{3}}$ layer. This layer communicates the mass flux between the Ekman layers and is consequently of order $E^{\frac{1}{2}}$. In order to derive the boundary conditions for this layer one needs the Ekman suction formula, which can be derived from (3.10a–c) and (3.3d). One finds for the antisymmetric case

$$\hat{w}_1(0,r) = \pm \frac{1}{2r\rho_{00}} \frac{\partial}{\partial r}\left[r\rho_{00}^{\frac{1}{2}}(1+\alpha^2 r^2)^{\frac{1}{4}} \left\{ \pm r - v_0(r, \pm 1) - \frac{r}{2}[\pm T_c - T_0(r, \pm 1)] \right\} \right] \quad \text{at} \quad z = \pm 1. \tag{A 20}$$

For the symmetric case, it can be shown that the lowest-order Ekman layers disappear. After calculating the net mass flux to be transported by the $E^{\frac{1}{3}}$ layer from (A 20), one finds that (A 10a) is to be replaced by

$$\tilde{\phi}_0(0,z) = r_0^2(1+\alpha^2 r_0^2)^{\frac{1}{4}} \left(1 - \frac{T_c(r_0)}{2}\right), \quad (z) < 1. \tag{A 21}$$

The temperature gradient at $\xi = 0$ has to be zero in this layer, which, according to (A 8), gives

$$\int_0^\infty \frac{\partial \tilde{\phi}_0}{\partial z}(\xi, z)\, d\xi = 0, \quad |z| < 1. \tag{A 22}$$

The boundary conditions for this $E^{\frac{1}{3}}$ layer are thus given by (A 21)–(A 22), (A 10b), (A 11) and (A 12). Because the swirl velocity in this layer is non-zero at $\xi = 0$, a geostrophic flow of order $E^{1/3}$ will occur. As in the previous cases, Ekman layer extensions will occur. To lowest order, however, no $E^{1/3}\times E^{1/3}$ regions appear to be present.

For the $E^{1/4}$ layer at an insulated vertical boundary, it can be shown that the boundary conditions for the boundary-layer solution are similar to the ones for the $E^{1/3}$ layer although the two layers are quite different. One consequently finds that the vertically averaged geostrophic swirl velocity field satisfies the kinematical boundary condition $\langle v_0\rangle = 0$, whereas the vertically averaged radial temperature gradient of the geostrophic flow is corrected by an $E^{1/4}$ layer of order $E^{1/4}$. The case discussed in § 3 is, in this respect, exceptional as the vertically averaged geostrophic temperature field happens to be zero everywhere. The boundary-value problems for the Ekman layer extensions and the $E^{1/4}\times E^{1/4}$ regions can also be derived in essentially the same way as for their counterparts in the $E^{1/3}$ layer case. Further details of such weak $E^{1/4}$ layers are given by Hultgren (1978).

The $E^{1/4}$ layer in the case discussed in § 4, i.e. a conducting vertical wall with a prescribed temperature distribution and insulated top and bottom walls, deserves some detailed comments. To calculate the flow in that layer, a stretched radial co-ordinate, η, as usual defined by

$$\eta = \frac{r_0 - r}{E^{1/4}}, \tag{A 23}$$

is to be used. The dependent variables, which are denoted by overbars, are assumed to possess asymptotic expansions of the form

$$(\bar{u}, \bar{v}, \bar{w}, \bar{p}, \bar{\rho}, \bar{T}) = E^s \sum_{n=0}^{N-1} E^{n/4}(\bar{u}_n, \bar{v}_n, \bar{w}_n, \bar{p}_n, \bar{\rho}_n, \bar{T}_n) + O(E^{N/4}). \tag{A 24}$$

From (2.4*a–f*), (A 23) and (A 24) one finds that

$$\bar{u}_0 = \bar{u}_1 = \bar{w}_0 = \bar{p}_0 = 0. \tag{A 25}$$

In addition to the Ekman layer extensions, one must also construct $E^{1/4}\times E^{1/4}$ regions of the kind discovered by Matsuda & Hashimoto (1978) in order to satisfy the boundary conditions at the top and bottom walls. In these $E^{1/4}\times E^{1/4}$ regions, the stretched axial co-ordinate θ, which is defined by

$$\theta = \frac{1 \pm z}{E^{1/4}}, \tag{A 26}$$

is used in addition to η. The dependent variables, which are denoted by double overbars, are assumed to possess asymptotic expansions of the form

$$(\bar{\bar{u}}, \bar{\bar{v}}, \bar{\bar{w}}, \bar{\bar{p}}, \bar{\bar{\rho}}, \bar{\bar{T}}) = E^s \sum_{n=0}^{N-1} E^{n/4}(\bar{\bar{u}}_n, \bar{\bar{v}}_n, \bar{\bar{w}}_n, \bar{\bar{p}}_n, \bar{\bar{\rho}}_n, \bar{\bar{T}}_n) + O(E^{N/4}). \tag{A 27}$$

From (2.4*a–f*), (A 26) and (A 27) it follows that

$$\bar{\bar{u}}_0 = \bar{\bar{u}}_1 = \bar{\bar{w}}_0 = \bar{\bar{w}}_1 = \bar{\bar{p}}_0 = \bar{\bar{p}}_1 = 0, \tag{A 28a}$$

$$\bar{\bar{v}}_0 = \frac{r_0 \bar{\bar{T}}_0}{2}, \tag{A 28b}$$

$$\frac{\partial^2 \bar{\bar{v}}_0}{\partial \eta^2} + \frac{\partial^2 \bar{\bar{v}}_0}{\partial \theta^2} = 0. \tag{A 28c}$$

The very close analogy between (A 15*b–c*) and (A 28*b–c*) should be noted. It means that the physical character of the flow in the $E^{1/3}\times E^{1/3}$ and $E^{1/4}\times E^{1/4}$ regions is the same, i.e. a degenerate form of a geostrophic flow.

Suppose now that there is a $E^{\frac{1}{4}}$ layer of order unity and that the axial and swirl velocities in this layer at the top and bottom are corrected by Ekman layer extensions of order unity. This means that the axial temperature gradient of order $E^{-\frac{1}{2}}$ in these Ekman layer extensions at the insulated top and bottom walls has to be corrected by $E^{\frac{1}{4}} \times E^{\frac{1}{4}}$ regions of order $E^{-\frac{1}{4}}$. Because the solution of (A 28*b*–*c*) can only fulfil one boundary condition at $\theta = 0$, $0 \leqslant \eta < \infty$, this gives a non-zero value, of order $E^{-\frac{1}{4}}$, of the swirl velocity for these values of θ and η. As there are no further correction fields available, this form of the expansion has to be discarded. If one tries to correct the swirl velocity at the top and bottom of an order unity $E^{\frac{1}{4}}$ layer with $E^{\frac{1}{4}} \times E^{\frac{1}{4}}$ regions of order unity and the axial temperature gradient in these regions by Ekman layer extensions of order $E^{\frac{1}{2}}$, one finds that the axial velocity of the $E^{\frac{1}{4}}$ layer cannot be corrected. This means that, for the class of limit solutions considered in the present work, an $E^{\frac{1}{4}}$ layer of order unity cannot exist at a conducting vertical boundary between insulated top and bottom walls. As a consequence, the boundary condition given by the averaged temperature distribution on such a vertical wall has to be satisfied by the geostrophic flow. There will, however, be an $E^{\frac{1}{4}}$ layer of order $E^{\frac{1}{2}}$ to handle the mass flux from the geostrophic flow. One finds that this weak $E^{\frac{1}{4}}$ layer has Ekman layer extensions of order $E^{\frac{1}{2}}$. There will also be $E^{\frac{1}{4}} \times E^{\frac{1}{4}}$ regions of order $E^{\frac{1}{4}}$ to correct the axial temperature gradient of the weak Ekman layer extensions at the top and bottom. This set of correction fields is very similar to that constructed by Matsuda & Hashimoto (1978). The remaining boundary condition to be taken care of by higher-order correction fields, which are not considered in this work, is a non-zero value of the swirl velocity of order $E^{\frac{1}{4}}$ at $z = \pm 1$, $\eta = O(1)$. On the length scale given by the size of the container, this uncorrected swirl velocity will appear as a concentrated ring source of angular momentum of strength $E^{\frac{1}{2}}$ and can therefore on reasonable grounds be neglected as far as the lowest order part of the geostrophic flow is concerned.

It should be pointed out that there may well be an $E^{\frac{1}{3}}$ layer of order unity in the case discussed in § 3. The reason for this is that the $E^{\frac{1}{3}}$ layer does not have to be connected to Ekman layer extensions of the same order as the $E^{\frac{1}{3}}$ layer itself [cf. the discussion in connexion with (A 13)–(A 16*c*)]. For the $E^{\frac{1}{4}}$ layer, however, this has to be the case.

Appendix 2. Computational method

The integro-partial-differential equation (3.20) and the non-homogeneous partial-differential equation (5.1), both subject to the appropriate boundary conditions, were solved by using finite difference schemes. It turns out to be somewhat more convenient to solve for the local angular velocity field, $\omega = v_0/r$, than for the swirl velocity field. With $r = (n-1)h$ and $z = (m-1)k$ the following finite difference approximations were utilized for the derivatives:

$$\frac{\partial \omega}{\partial r} = \frac{\omega(r+h,z)-\omega(r-h,z)}{2h}, \qquad \text{(B 1a)}$$

$$\frac{\partial^2 \omega}{\partial r^2} = \frac{\omega(r+h,z)-2\omega(r,z)+\omega(r-h,z)}{h^2}, \qquad \text{(B 1b)}$$

$$\frac{\partial^2 \omega}{\partial z^2} = \frac{\omega(r,z+k)-2\omega(r,z)+\omega(r,z-k)}{k^2}, \qquad \text{(B 1c)}$$

and the integral in (3.20) was evaluated by means of the trapezoidal rule. Equations (3.20) and (5.1) become

$$a_n \frac{\omega_{n+1,m}-2\omega_{n,m}+\omega_{n-1,m}}{h^2}+b_n\frac{\omega_{n+1,m}-\omega_{n-1,m}}{2h}+a_n\frac{\omega_{n,m+1}-2\omega_{n,m}+\omega_{n,m-1}}{k^2}$$
$$= k\sum_{j=1}^{M+1}[1-\tfrac{1}{2}(\delta_{i,j}+\delta_{M+1,j})]\left(r_n\frac{\omega_{n+1,j}-2\omega_{n,j}+\omega_{n-1,j}}{h^2}+\frac{\omega_{n+1,j}-\omega_{n-1,j}}{2h}\right),$$
$$n = 2, N, \quad m = 2, M \qquad \text{(B 2)}$$

and

$$a_n \frac{\omega_{n+1,m}-2\omega_{n,m}+\omega_{n-1,m}}{h^2}+b_n\frac{\omega_{n+1,m}-\omega_{n-1,m}}{2h}$$
$$+a_n\frac{\omega_{n,m+1}-2\omega_{n,m}+\omega_{n,m-1}}{k^2} = N_n, \quad n = 2, N, \quad m = 2, M, \qquad \text{(B 3)}$$

respectively, where

$$\omega_{n,m} = \omega[(n-1)h, (m-1)k], \qquad \text{(B 4}a)$$
$$a_n = r_n(1+\alpha^2 r_n^2), \qquad \text{(B 4}b)$$
$$b_n = 1+3\alpha^2 r_n^2, \qquad \text{(B 4}c)$$
$$r_n = (n-1)h, \qquad \text{(B 4}d)$$
$$N_n = \frac{1}{2}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{v_+ + v_-}{r}-\tfrac{1}{2}(T_+ + T_-)\right]\right\} \quad \text{at} \quad r = (n-1)h,\ z = (m-1)k, \qquad \text{(B 4}e)$$

and $\delta_{i,j}$ is the Kronecker delta.

Equations (B 2) and (B 3) hold for all interior points. The remaining $2(M+N)$ equations needed in each case are obtained from the boundary conditions. In particular, it is easily shown that $\partial\omega/\partial r = 0$ at $r = 0$, i.e.

$$\omega_{1,m} = \omega_{2,m}, \quad m = 2, M. \qquad \text{(B 5)}$$

In the actual calculations the stepsizes $h = 0{\cdot}05$ and $k = 0{\cdot}1$ were chosen. The accuracy of each calculation was controlled by a subsequent calculation using twice the stepsize in each direction.

The accuracy of the finite difference scheme was, for the antisymmetric case, compared with a series solution obtained by separation of variables of the same kind as the solution given by Bark & Bark (1978) for finite but otherwise arbitrary Mach numbers. Such solutions converge rapidly in the antisymmetric case and a four-digit agreement was found. For the symmetric case, however, the series solution obtained by separation of variables did not converge.

REFERENCES

BARCILON, V. & PEDLOSKY, J. 1966 Linear theory of rotating stratified fluid motions. *J. Fluid Mech.* **29**, 1.

BARK, F. H. & BARK, T. H. 1976 On vertical boundary layers in a rapidly rotating gas. *J. Fluid Mech.* **78**, 749.

BARK, F. H. & BARK, T. H. 1978 On axially antisymmetric geostrophic flows in rapidly rotating gases. To be published in *J. Méc.*

BARK, F. H., MEIJER, P. S. & COHEN, H. I. 1978 Spin up of a rapidly rotating gas. *Phys. Fluids*, **21**, 531.

HASHIMOTO, K. 1977 Flows in a rapidly rotating cylinder. Ph.D. thesis. Department of Aeronautical Engineering, Kyoto University.

HULTGREN, L. S. 1978 Stability of axisymmetric gas flows in a rapidly rotating cylindrical container. Ph.D. thesis, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology.

MATSUDA, T. 1977 On the importance of the thermal boundary condition to the flow in a strongly rotating fluid. *Proc. Sec. Workshop on Gases in Strong Rotation* (ed. Soubbaramayer), p. 77. Centre d'Etudes Nucléaires de Saclay.

MATSUDA, T. & HASHIMOTO, K. 1976 Thermally, mechanically or externally driven flows in a gas centrifuge with insulated horizontal end plates. *J. Fluid Mech.* **78**, 337.

MATSUDA, T. & HASHIMOTO, K. 1978 The structure of the Stewartson layers in a gas centrifuge. Part 1. Insulated end plates. *J. Fluid Mech.* **85**, 433.

MATSUDA, T., HASHIMOTO, L. & TAKEDA, H. 1976 Thermally driven flow in a gas centrifuge with an insulated side wall. *J. Fluid Mech.* **73**, 389.

MATSUDA, T. & TAKEDA, H. 1978 The structure of the Stewartson layers in a gas centrifuge. Part 2. Insulated side wall. *J. Fluid Mech.* **85**, 443.

SAKURAI, T. & MATSUDA, T. 1974 Gas dynamics of a centrifugal machine. *J. Fluid Mech.* **62**, 727.

SOUBBARAMAYER 1977 *Proc. Sec. Workshop on Gases on Strong Rotation.* Centre d'Etudes Nucléaires de Saclay.

STEWARTSON, K. 1957 On almost rigid rotations. *J. Fluid Mech.* **3**, 16.